Hamiltonian formalism in Friedmann cosmology and its quantization

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We propose a Hamiltonian formalism for a generalized Friedmann-Roberson-Walker cosmology model in the presence of both a variable equation of state (EOS) parameter $w(a)$ and a variable cosmological constant $\Lambda(a)$, where a is the scale factor. This Hamiltonian system containing 1 degree of freedom and without constraint, gives Friedmann equations as the equation of motion, which describes a mechanical system with a variable mass object moving in a potential field. After an appropriate transformation of the scale factor, this system can be further simplified to an object with constant mass moving in an effective potential field. In this framework, the Λ cold dark matter model as the current standard model of cosmology corresponds to a harmonic oscillator. We further generalize this formalism to take into account the bulk viscosity and other cases. The Hamiltonian can be quantized straightforwardly, but this is different from the approach of the Wheeler-DeWitt equation in quantum cosmology.

I. INTRODUCTION

Since the current accelerating expansion of our Universe was discovered [1] around 1998 and 1999, theoretical physicists have devoted increasingly more attention to the Friedmann-Roberson-Walker (FRW) model as a standard framework in cosmology study. The Λ cold dark matter (ΛCDM) model as the standard model of cosmology so far fits well with observational data whereas it has had some serious theoretical problems. To make a comparison to the ΛCDM model, physicists have built many cosmological models that are able to give out the effective Friedmann equations with variable cosmological constant. To quantize the Friedmann equations, the commonly used theory is the Wheeler-DeWitt equation [2], which has been studied and applied widely in quantum cosmology [3]. Starting from the Hilbert-Einstein action with the Roberson-Walker (RW) metric, the corresponding Hamiltonian $\mathcal{H}$ can be obtained. Then the Friedmann equation plays the role as the constraint $\mathcal{H} = 0$, which leads to the Wheeler-DeWitt equation. In the present work, we consider the Friedmann equations as basic equations and find a Hamiltonian system that gives Friedmann equations as classical equations of motion without constraint.

Many Ansätze of the variable cosmological constant have been studied in the literature, such as Refs. [4, 5, 6, 7]. Moreover, some models motivated from the string theory give an effective cosmological term when reduced to the FRW framework. We assume that the equation of state (EOS) parameter $w \equiv p/\rho$ can also be variable, which means that the contents of the Universe, except the cosmological term, are generalized to a nonperfect

fluid, or perfect fluid as a special case. In observational cosmology, the redshift z is regarded as an observable quantity and related to the scale factor a by $z \equiv a_0/a - 1$. Therefore, we investigate a general case that both the EOS parameter w and the cosmological constant Λ can be functions of the scale factor a, and take into account the bulk viscosity.

As an extension of the problem, we construct a Hamiltonian formalism for a system described by the following equation:

$$\ddot{q} = f_1(q)\dot{q}^2 + \eta\dot{q} + f_2(q),$$

where $f_1(q)$ and $f_2(q)$ are arbitrary functions, and η is constant. Also it can be regarded as a generalization of the damping harmonic oscillator. The corresponding Hamiltonian describes an object with variable mass moving in a potential field. After an appropriate canonical transformation, this system can be further simplified to an object with constant mass moving in an effective potential field. Thus, differential models in the FRW framework are characterized by their effective potentials. This is a general formalism and it can be applied to many cosmological models, for example, that the ΛCDM model corresponds to a harmonic oscillator. Since the quantization of Friedmann equations can provide an insight to quantum cosmology as a glimpse of quantum gravity, we also make some remarks on the quantum case, which provides a correspondence between cosmology and quantum mechanics.

The paper is organized as follows. In Sec. II we present a generalized FRW model and the corresponding Hamiltonian to describe the Friedmann equations. Then we find a canonical transformation to further simplify the problem, and give some examples and special cases. In Sec. III we show that our framework can also be applied in the dissipative case with bulk viscosity. In Sec. IV we turn our attention to the relation to the observable

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quantities and review some issues of the Bianchi identity. In Sec. V we make some remarks on quantum cosmology from our approach. In the last section we present the conclusion and discuss some future subjects.

II. HAMILTONIAN FORMALISM

A. Hamiltonian description of the Friedmann equations

We consider the RW metric in the flat space geometry ($k=0$) as the case favored by current cosmic observational data:

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2d\Omega^2), \qquad (1)$$

where $a(t)$ is the scale factor. The energy-momentum tensor for the cosmic fluid can be written as

$$\tilde{T}_{\mu\nu} = (\rho + p)U_\mu U_\nu + (p + \rho_\Lambda)g_{\mu\nu}, \qquad (2)$$

where $\rho_\Lambda = \Lambda/(8\pi G)$ is the energy density of the cosmological constant. Thus, Einstein's equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\tilde{T}_{\mu\nu}$ contains two independent equations:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \qquad (3a)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \qquad (3b)$$

The EOS of the matter (cosmic fluid except the cosmological constant) is commonly assumed to be

$$p = (\gamma - 1)\rho. \qquad (4)$$

Cosmologists usually call Eq. (3a) as the Friedmann equation and Eq. (3b) as the acceleration equation in the literature, whereas for simplicity we name both Eqs. (3a) and (3b) Friedmann equations here. For generality, we assume that both γ and Λ are functions of the scale factor a, thus we call it the generalized FRW model. Combining the Friedmann equations with the EOS, we obtain

$$\frac{\ddot{a}}{a} = -\frac{3\gamma(a) - 2}{2}\frac{\dot{a}^2}{a^2} + \frac{\gamma(a)\Lambda(a)}{2}, \qquad (5)$$

which determines the evolution of the scale factor.

We regard Eq. (5) as a basic starting point; therefore, if the dynamical equation for the scale factor can be written as that form, the present framework can be valid. If the Newton constant G is constant and the cosmological constant Λ is variable, the energy-momentum tensor for the matter cannot individually conserved [5, 6], which implies an interaction between the matter and vacuum energy. In the following, we assume G to be constant until Sec. IV.

Our aim is to find a Hamiltonian description of Eq. (5) as the classical equation of motion. We start from the following Lagrangian

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2}M(q)\dot{q}^2 - V(q), \qquad (6)$$

and the corresponding Hamiltonian thus is

$$\mathcal{H}(q, p) = \frac{p^2}{2M(q)} + V(q), \qquad (7)$$

with the canonical Poisson bracket $\{q, p\} = 1$. One can check that the equation of motion for Eq. (6) or (7) is

$$\frac{\ddot{q}}{q} = -\frac{1}{2}\frac{\partial \ln M}{\partial \ln q}\frac{\dot{q}^2}{q^2} - \frac{1}{Mq}\frac{\partial V}{\partial q}. \qquad (8)$$

This equation possesses the same form as Eq. (5). Therefore, by comparing Eq. (5) with Eq. (8), we can take a as the general coordinate and solve the functions $M(a)$ and $V(a)$. Then the Lagrangian $\mathcal{L} = \frac{1}{2}M(a)\dot{q}^2 - V(a)$ with

$$M = \exp\left(\int \frac{3\gamma - 2}{a}da\right), \quad V = -\frac{1}{2}\int M\gamma\Lambda a\,da, \quad (9)$$

gives Eq. (5) as the equation of motion. For some specified functions $\gamma = \gamma(a)$ and $\Lambda = \Lambda(a)$, the above integrations can be evaluated out to give $M(a)$ and $V(a)$ explicitly.

Now we can see that the generalized FRW model essentially corresponds to an object with variable mass $M(a)$ moving in a potential field $V(a)$. In the following, we will show that this picture can be further simplified as an object with constant mass moving in an effective potential field $\tilde{V}(\phi)$, after an appropriate transformation of the scale factor.

B. Canonical transformation

The above problem can be generalized as the Hamiltonian description of the nonlinear equation

$$\ddot{q} = f_1(q)\dot{q}^2 + f_2(q), \qquad (10)$$

where $f_1(q)$ and $f_2(q)$ are two specified functions. This equation can be derived by the Lagrangian $\mathcal{L} = \frac{1}{2}M(q)\dot{q}^2 - V(q)$ with

$$M = \exp\left(-2\int f_1(q)dq\right), \quad V = -\int Mf_2(q)dq. \quad (11)$$

We define a new variable ϕ as (see Appendix)

$$\phi \equiv \int \exp\left(-\int f_1(q)dq\right)dq. \qquad (12)$$

This transformation can eliminate the $\dot{q}^2$ term and gives the equation for the variable ϕ as in

$$\ddot{\phi} = f_2(q)\exp\left(-\int f_1(q)dq\right)\Big|_{q\to\phi}, \qquad (13)$$

where $q \to \phi$ denotes using Eq. (12) to change the variable q to ϕ. Since there is no $\dot{\phi}^2$ term in Eq. (13), this can be regarded as a partial linearization. Therefore, the system of Eq. (10) transformed to Eq. (13) can be described by the Lagrangian

$$\mathcal{L}(\phi, \dot{\phi}) = \frac{1}{2}\dot{\phi}^2 - \tilde{V}(\phi), \qquad (14)$$

with the potential as

$$\begin{aligned}
\tilde{V}(\phi) &= -\int \left[f_2(q) \exp\left(-\int f_1(q) dq \right) \right]_{q \to \phi} d\phi \\
&= -\int f_2(q) \exp\left(-2\int f_1(q) dq \right) dq \bigg|_{q \to \phi} . \quad (15)
\end{aligned}$$

The simplification of the problem by Eq. (12) is essentially the canonical transformation

$$q \to \phi, \quad p_q \to p_\phi, \quad \mathcal{H}(q, p_q) \to \mathcal{H}(\phi, p_\phi), \qquad (16)$$

where $p_q = M(q)\dot{q}$, $p_\phi = \dot{\phi}$, and $\mathcal{H}(\phi, p_\phi) = \frac{1}{2}p_\phi^2 + \tilde{V}(\phi)$. Therefore, the classical and quantum properties of different models are characterized by the effective potentials.

For Eq. (5) as a special case, the new variable ϕ is given by

$$\phi = \int \exp\left(\int \frac{3\gamma - 2}{2a} da \right) da. \qquad (17)$$

C. Some examples

We will give some special cases of the above general framework to show some applications. If both γ and Λ are constant for a simple case, the integrations in Eq. (17) can be evaluated out as

$$\phi = \frac{2}{3\gamma} a^{3\gamma/2}, \quad \gamma \neq 0, \qquad (18a)$$

$$= \ln a, \quad \gamma = 0. \qquad (18b)$$

Now we consider $\gamma \neq 0$ for example. The special case $\gamma = 1$ corresponds to the ΛCDM model. The equation for ϕ can be obtained as $\ddot{\phi} - \frac{3}{4}\gamma^2\Lambda\phi = 0$, and the corresponding Lagrangian is

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{3}{8}\gamma^2\Lambda\phi^2. \qquad (19)$$

We can see that the simplest model in cosmology just corresponds to a harmonic oscillator after linearization. In particular, this is a upside-down harmonic oscillator for the asymptotic de Sitter Universe.

We can add the curvature effect to the ΛCDM model, which is described by the special case $m = 2$ of the following equation:

$$\frac{\ddot{a}}{a} = -\frac{3\gamma - 2}{2}\frac{\dot{a}^2}{a^2} + \frac{\gamma\Lambda}{2} - \frac{k}{a^m}. \qquad (20)$$

Here the parameters γ, Λ, and m are all constants. This equation possesses the same form of Eq. (10). By defining ϕ as Eq. (12) and using Eq. (15), we obtain the effective potential as

$$\tilde{V}(\phi) = -\frac{3}{8}\gamma^2\Lambda\phi^2 + \frac{k}{3\gamma - m}\left(\frac{3\gamma\phi}{2} \right)^{2 - 2m/3\gamma}, \qquad (21)$$

for $\gamma \neq 0$ and $m \neq 3\gamma$.

Another example is the Friedmann equations during the inflation era. In the study of inflation, we usually use the conformal time τ instead of the comoving cosmic time t. Here we assume that a constant term $-p_0$ is in the EOS during inflation. Thus the Friedmann equations combined with the EOS $p = -\rho - p_0$ yield

$$\frac{a''}{a^3} = 2\frac{a'^2}{a^4} + \frac{\kappa^2}{2}p_0, \qquad (22)$$

where the prime denotes a derivative with respect to τ, and $\kappa^2 = 8\pi G$. By defining $\phi = -1/a$, the equation for ϕ is $\phi''\phi + (\kappa^2/2)p_0 = 0$. The effective potential is thus

$$\tilde{V}(\phi) = \frac{\kappa^2}{2}p_0 \ln |\phi|. \qquad (23)$$

Moreover, if we add the curvature term in this case, it corresponds to a ϕ^2 potential.

III. BULK VISCOSITY

We assume that the cosmic fluid possesses some dissipation effects. Since the sheer tensor $\sigma_{\mu\nu} = 0$ for RW metric, the sheer viscosity does not contribute to the evolution in Friedmann cosmology. The energy-momentum tensor for nonperfect fluid concerning bulk viscosity in the right-hand side of Einstein's equation is given by [8, 9]

$$T_{\mu\nu} = \rho U_\mu U_\nu + (p - \zeta_0\theta)h_{\mu\nu}, \qquad (24)$$

where $h_{\mu\nu} \equiv g_{\mu\nu} + U_\mu U_\nu$ is the projection operator, $\theta \equiv U^\alpha_{;\alpha} = 3\dot{a}/a$ is the scalar expansion, and ζ is the bulk viscosity coefficient. Consequently, Eq. (5) should be modified as

$$\frac{\ddot{a}}{a} = -\frac{3\gamma(a) - 2}{2}\frac{\dot{a}^2}{a^2} + 12\pi G\zeta_0\frac{\dot{a}}{a} + \frac{\gamma(a)\Lambda(a)}{2}. \qquad (25)$$

where both γ and Λ can be functions of a for generality, and ζ_0 is constant. We also find a Hamiltonian

$$\mathcal{H}(a, p_a, t) = \frac{p_a^2}{2M(a, t)} + V(a, t), \qquad (26)$$

with the Poisson bracket $\{a, p_a\} = 1$ to give Eq. (25) as the classical equation of motion. The functions in this Hamiltonian are given by

$$M = \exp\left(\int \frac{3\gamma - 2}{a} da - 12\pi G\zeta_0 t \right), \qquad (27a)$$

$$V = -\frac{1}{2}\int M\gamma\Lambda a \, da. \qquad (27b)$$

Although a dissipative system cannot be described by a conservative Hamiltonian generally, one can directly check that the classical equation of motion for the Hamiltonian Eq. (26) is Eq. (25). As a special case, the equation for a damping harmonic oscillator can be derived by the Caldirora-Kani (CK) Hamiltonian [10].

The above problem can be generalized to construct a Hamiltonian system for the equation

$$\ddot{q} = f_1(q)\dot{q}^2 + \eta\dot{q} + f_2(q), \tag{28}$$

where η is constant. It can be derived by the Hamiltonian $H(q, p, t) = \frac{1}{2}M(q, t)^{-1}p^2 + V(q, t)$ with

$$M = \exp\left(-2\int f_1(q)dq - \eta t\right), \quad V = -\int Mf_2(q)dq. \tag{29}$$

Similarly, by using the new variable ϕ defined by Eq. (12), the equation for ϕ is

$$\ddot{\phi} = \eta\dot{\phi} + f_2(q)\exp\left(-\int f_1(q)dq\right)\bigg|_{q\to\phi}. \tag{30}$$

Now we consider a very special case that both γ and Λ are constant; then ϕ defined by Eq. (18a) satisfies

$$\ddot{\phi} - 12\pi G\zeta_0\dot{\phi} - \frac{3}{4}\gamma^2\Lambda\phi = 0, \tag{31}$$

which describes a damping harmonic oscillator.

The damping harmonic oscillator

$$M\ddot{q} = -\eta\dot{q}(t) - \frac{\partial V(q)}{\partial q}, \tag{32}$$

has been studied in quantum mechanics. The CK Hamiltonian

$$H = \frac{1}{2M}e^{-\eta t/M}p^2 + \frac{1}{2}M\omega^2 e^{\eta t/M}q^2, \tag{33}$$

with the commutation relation $[q, p] = i\hbar$, can yield the dissipation equation (32) through the Heisenberg equation [10]. Our work can be regarded as a generalization to the case of variable mass. It is the variable mass that generates a nonlinear term in the equation of motion that describes the generalized FRW model.

In our previous work [9], we have proposed an EOS as

$$p = (\gamma - 1)\rho - \frac{2}{\sqrt{3}\kappa T_1}\sqrt{\rho} - \frac{2}{\kappa^2 T_2^2} \tag{34}$$

where the parameters γ, T_1 and T_2 are constants. Combining the Friedmann equations with this more practical EOS, we obtain the dynamical evolution equation for the scale factor as

$$\frac{\ddot{a}}{a} = -\frac{3\gamma - 2}{2}\frac{\dot{a}^2}{a^2} + \frac{1}{T_1}\frac{\dot{a}}{a} + \frac{1}{T_2^2}. \tag{35}$$

This model possesses a large variety of properties, such as that we have found a scalar field model which is equivalent to the above EOS. For related works on the modified EOS, see Ref. [9, 11, 12, 13]. The present work can also be regarded as a generalization of the EOS to $\gamma = \gamma(a)$ and $T_2 = T_2(a)$. And the corresponding Hamiltonian formalism for this system can be constructed similarly.

IV. RELATIONS TO THE OBSERVABLE QUANTITIES

The observations of the supernovae (SNe) Ia have provided the direct evidence for the cosmic accelerating expansion of our current Universe [1]. A bridge between the cosmological theory and the observation data is the H-z relation, where $H \equiv \dot{a}/a$ is the Hubble parameter and z is the redshift. For example, the ΛCDM model in cosmology can be described mainly as $H^2(z) = H_0^2[\Omega_m(1 + z)^3 + 1 - \Omega_m]$, where Ω_m is the matter energy density. This model fits the observational data well and provides the cosmological constant as the simplest candidate for dark energy. In a sense, the different cosmological models are characterized by the corresponding H-z relations.

There is also a systematic way to construct the Hamiltonian starting from the general model

$$H^2 = f(a), \tag{36}$$

where $f(a)$ is a specific function of the scale factor a, according to the model. By differentiating Eq. (36), we obtain that it is a solution of the following equation:

$$\frac{\ddot{a}}{a} = -\frac{3\gamma - 2}{2}\frac{\dot{a}^2}{a^2} + \frac{3\gamma f(a)}{2} + \frac{af'(a)}{2}, \tag{37}$$

which possesses the same form of Eq. (5) or (10). The corresponding coefficients are given by

$$f_1(a) = -\frac{3\gamma - 2}{2a}, \quad f_2(a) = \frac{3\gamma af(a)}{2} + \frac{a^2 f'(a)}{2}. \tag{38}$$

Then by applying Eq. (11) we can obtain the corresponding Hamiltonian. Therefore, even if the EOS for a cosmological model is not explicitly linear in ρ, the Hamiltonian formalism in the present work can also be applied if the effective Friedmann equation $H^2 = f(a)$ can be given out for that model.

Many approaches such as modified gravity [14] can be reduced to effective Friedmann equations in the form $H^2 = f(a)$. Since ΛCDM model fits the SNe Ia data well, the reasonable cosmological models should be reduced to Friedmann cosmology in an effective way and give out the right H-z relation, in order to make a comparison with the ΛCDM model. In our case, the Friedmann equations in terms of the Hubble parameter can be written as

$$aH\frac{dH}{da} = -\frac{3\gamma}{2}H^2 + \tilde{\Lambda}(a). \tag{39}$$

Here γ is assumed to be constant for simplicity. This equation is linear in H^2 and the effective term $\tilde{\Lambda}(a)$ is an inhomogeneous term. The solution in terms of $H(z)$ concerning the initial condition $H(0) = H_0$ is given out by

$$H(z)^2 = H_0^2(1 + z)^{3\gamma}\left[1 - 2\int_0^z \tilde{\Lambda}(z')(1 + z')^{-3\gamma-1}dz'\right]. \tag{40}$$

In the power-law ΛCDM model, the contributions of different components are separated in H^2, such as a constant for the cosmological constant, and a $(1+z)^2$ factor for the curvature term. But in the general case, the contribution of the matter cannot be separated from the above solution. This problem is related to the conservation law of the matter, which has been investigated in Refs. [5, 6].

The Bianchi identity for the energy-momentum tensor Eq. (2) gives

$$\dot{\rho}_\Lambda + \dot{\rho} + 3H(\rho + p) = 0, \tag{41}$$

which implies that energy transfer will exist between the matter and the vacuum energy. An intuitive idea has been proposed that if both G and Λ are variable, the ordinary energy-momentum tensor can be individually conserved, i.e., $\dot{\rho}+3H(\rho+p)=0$ [6]. This is achieved by combining the Bianchi identity for the variable G and Λ model

$$\frac{d}{dt}[G(\rho_\Lambda + \rho)] + 3GH(\rho + p) = 0, \tag{42}$$

with the following constraint:

$$(\rho + \rho_\Lambda)\dot{G} + G\dot{\rho}_\Lambda = 0. \tag{43}$$

The authors of Ref. [6] assume that both the Newton constant G and the cosmological constant Λ are functions of a scale parameter μ and apply the renormalization group approach to cosmology. If $G(\mu)$ evolves by a logarithmic law and $\rho_\Lambda(\mu)$ evolves quadratically with μ, then this picture can explain the evolution of the Universe, and at the same time, the variable G can explain the flat rotation curves of the galaxies without introducing the dark matter hypothesis.

V. REMARKS ON QUANTUM COSMOLOGY

We have obtained a classical Hamiltonian formalism of the Friedmann equations. Generally, once a Hamiltonian is obtained, the system can be quantized straightforwardly by replacing the Poisson bracket with the commutation relation $[q, p] = i$. However, we need to take into account the ambiguity in the ordering of noncommuting operators q and p. For simplicity, we ignore the ordering ambiguity here. In terms of the new variable ϕ, the corresponding Schrödinger's equation can be written as

$$\mathcal{H}(\phi, \hat{p}_\phi)\Psi(\phi) = E\Psi(\phi), \tag{44}$$

where $\hat{p}_\phi = -i\partial_\phi$. To make a comparison between our approach and the Wheeler-DeWitt equation, we only take the ΛCDM model as a very special case for an illustrative example. The corresponding Hamiltonian for Eq. (19) in the case $\gamma = 1$ is

$$\mathcal{H} = \frac{1}{2a}p^2 - \frac{1}{6}\Lambda a^3, \tag{45}$$

where $p = a\dot{a}$. In the approach of the Wheeler-DeWitt equation, $\mathcal{H} = 0$ is a constraint [2, 15], thus the quantization gives $(\partial_a^2 + \frac{\Lambda}{3}a^4)\Psi(a) = 0$. This is an anharmonic oscillator with zero energy eigenvalue. In our case, the Hamiltonian is nonzero and proportional to the matter energy density, which we show in the following. The solution of Eq. (39) with $\tilde{\Lambda}(a) = \Lambda/2$ is

$$
\begin{aligned}
H^2 &= \left(H_0^2 - \frac{\Lambda}{3}\right)a^{-3} + \frac{\Lambda}{3} \\
&= H_0^2[\Omega_m a^{-3} + 1 - \Omega_m], \tag{46}
\end{aligned}
$$

where $\Omega_m \equiv 1 - \Lambda/(3H_0^2)$. Therefore, the Hamiltonian can be calculated as

$$\mathcal{H} = \frac{a^3}{2}\left(\frac{\dot{a}^2}{a^2} - \frac{\Lambda}{3}\right) = \frac{1}{2}H_0^2\Omega_m. \tag{47}$$

After a canonical transformation by Eq. (16), the Schrödinger's equation in terms of ϕ becomes

$$\left[-\frac{1}{2}\frac{d^2}{d\phi^2} - \frac{3}{8}\Lambda\phi^2\right]\Psi(\phi) = E\Psi(\phi). \tag{48}$$

Thus, for the asymptotic de Sitter Universe, the ΛCDM model corresponds to an upside-down harmonic oscillator in our formalism. Such an oscillator also appears in the matrix description of de Sitter gravity [16].

We can transform the de Sitter Universe to the dual anti-de Sitter Universe by employing the scale factor duality [17], which has been found that $a \to a^{-1}$ gives $H \to -H$ and other consequences. The duality for Eq. (5) is given by

$$a \to a^{-1}, \quad \gamma \to -\gamma, \quad \Lambda \to -\Lambda, \quad \phi \to -\phi. \tag{49}$$

It can be checked easily that Eq. (5) is invariant under these transformations. If we use the dual scale factor a^{-1}, the corresponding potential becomes $\tilde{V}(\phi) = +\frac{3}{8}\gamma^2\Lambda\phi^2$. In fact, quantization in de Sitter spacetime is one of the major difficulties of string theory at one time (though this picture has changed a little bit after Kachru-Kallosh-Linde- Trivedi theory appeared). It seems that quantizing de Sitter cosmology is no difference, since the time variable used is the same, and it is known that there is no global timelike coordinates in de Sitter spacetime. Some quantum effects of a scalar field in de Sitter background can be found in Ref. [18].

VI. CONCLUSION AND DISCUSSION

We have proposed a systematic scheme to describe the Friedmann equations through a Hamiltonian formalism. The generalized FRW model accompanied by both variable EOS parameter and variable cosmological constant admits a Hamiltonian description without constraint. After an appropriate canonical transformation, the system can be significantly simplified to an object moving in

an effective potential field. The bulk viscosity can also be taken into account by a time-dependent Hamiltonian. Some examples are given explicitly, such as the ΛCDM model, the curvature term effect, and the inflation period. The quantization of the system provides a new approach to study the potential quantum cosmology, which is an intriguing topic in theoretical physics research.

We shall discuss some possible future developments of our work. As we have claimed, the formalism in this work can be applied to a large variety of cosmological models. By solving the Schrödinger equation $\mathcal{H}(\phi, \hat{p}_\phi)\Psi = E\Psi$, the cosmological wave function can be obtained for a specific model. Here we consider the curvature effect, for example, which is described by the potential Eq. (21) with parameters $\Lambda = 0$, $\gamma = 1$, and $m = 2$. The corresponding Schrödinger equation can be solved in terms of the biconfluent Heun equation (BHE) [19]. We can also start from the effective Lagrangian and study the observational effects when we modify the potential. We believe that our formalism would give a new perspective to the potential study of quantum cosmology physics.

APPENDIX A: MATHEMATICAL NOTES

A more general correspondence between a Hamiltonian and its equation of motion is given in Ref. [19]. The equation of motion of the Hamiltonian

$$H(q, p, t) = \frac{1}{f(t)}\left(P_0(q,t)p^2 + P_1(q,t)p + P_2(q,t)\right),$$
(A1)

is given out by

$$\ddot{q} = \frac{1}{2}\frac{\partial \ln P_0}{\partial q}\frac{\dot{q}^2}{q^2} - \left(\frac{\partial \ln f}{\partial t} - \frac{\partial \ln P_0}{\partial t}\right)\dot{q} + \frac{P_0}{f^2}\left(\frac{\partial}{\partial q}\frac{P_1^2}{2P_0} + f\frac{\partial}{\partial t}\frac{P_1}{P_0} - 2\frac{\partial V}{\partial q}\right).$$
(A2)

In the mathematical aspect, Eq. (28) can be further generalized to the following equation:

$$\ddot{q} = F_1(q,t)\dot{q}^2 + F_2(q,t)\dot{q} + F_3(q,t),$$
(A3)

however, here the coefficients F_1 and F_2 are not completely independent. Comparing with Eq. (A2), we can see that the condition $2\partial_t F_1(q,t) = \partial_q F_2(q,t)$ must be satisfied for consistency. In the present work, both $f_1(q)$ and η have safely satisfied this condition.

We shall explain why we choose the transformation as in Eq. (12). Starting from the following equation

$$\ddot{q} = f_1(q)\dot{q}^2 + \eta\dot{q} + f_2(q),$$
(A4)

we expect that after an appropriate change of variable $\phi(q)$, the above equation can be transformed as

$$\ddot{\phi} = \eta\dot{\phi} + g(\phi).$$
(A5)

By differentiating $\phi(q)$, we obtain $\dot{\phi} = \phi'\dot{q}$, and $\ddot{\phi} = \phi''\dot{q}^2 + \phi'\ddot{q}$, where the prime denotes a derivative with respect to q. Substituting ϕ, $\dot{\phi}$, and $\ddot{\phi}$ into Eq. (A5), we obtain

$$\ddot{q} = -\frac{\phi''}{\phi'}\dot{q}^2 + \eta\dot{q} + \frac{g}{\phi'}$$
(A6)

Now it turns out that by defining $-\phi''/\phi' = f_1(q)$, which can be solved as the form Eq. (12), the $\dot{q}^2$ term can be eliminated.

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